[1] Consider the following linear recurrence equation.

$$x_{n+2} = 5x_{n+1} - 6x_n \quad (n = 0, 1, 2 \cdots)$$

[1-1] When $\boldsymbol{v}_{n+1} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix}$, the relationship among x_n , x_{n+1} , and x_{n+2} can be written as $\boldsymbol{v}_{n+1} = \boldsymbol{A}\boldsymbol{v}_n$ using a second-order square matrix $\boldsymbol{A}$. When $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$, determine a and b .

[1-2] $\boldsymbol{A}$ can be transformed as $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1}$ using a second-order regular matrix $\boldsymbol{P}$, its inverse matrix $\boldsymbol{P}^{-1}$, and a diagonal matrix $\boldsymbol{B}$. Determine $\boldsymbol{P}$, $\boldsymbol{P}^{-1}$, and $\boldsymbol{B}$.

[1-3] Based on the relationship, $\boldsymbol{v}_{n+1} = \boldsymbol{A}\boldsymbol{v}_n$, express $\boldsymbol{v}_n$ using $\boldsymbol{A}$ and $\boldsymbol{v}_0$.

[1-4] Using the relationship between $\boldsymbol{v}_n$ and $\boldsymbol{v}_0$ obtained in [1-3], express x_n using x_0 and x_1 .

[2] Answer the following questions related to the Laplace transform. With a complex number s , the Laplace transform $F(s)$ of a function $f(t)$ is expressed as follows.

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt \qquad (\mathrm{Re}\,[s] > \alpha) \tag{1}$$

The inverse Laplace transform $f(t)$ of $F(s)$ is expressed as

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, e^{st} ds \qquad (c > \alpha) \tag{2}$$

where c and α are real numbers, $i^2 = -1$ and $\mathrm{Re}\,[s] > \alpha$ is the region of convergence.

[2-1] Calculate the Laplace transform $F(s)$ of the following function $f(t)$. Note that the region where $\mathrm{Re}\,[s] > 1$ is considered.

$$f(t) = \begin{cases} e^t & (t \geq 0) \\ 0 & (t < 0) \end{cases}$$

[2-2] Calculate the convolution integral $g(t)$ of the functions $f(t)$ and $f(t)$ defined in [2-1]. The convolution integral is expressed as follows.

$$g(t) = f(t) * f(t) = \int_{-\infty}^{\infty} f(\tau)\, f(t-\tau)\, d\tau$$

[2-3] Calculate the Laplace transform $G(s)$ of the function $g(t)$ obtained from [2-2]. Note that the region where $\mathrm{Re}\,[s] > 1$ is considered.

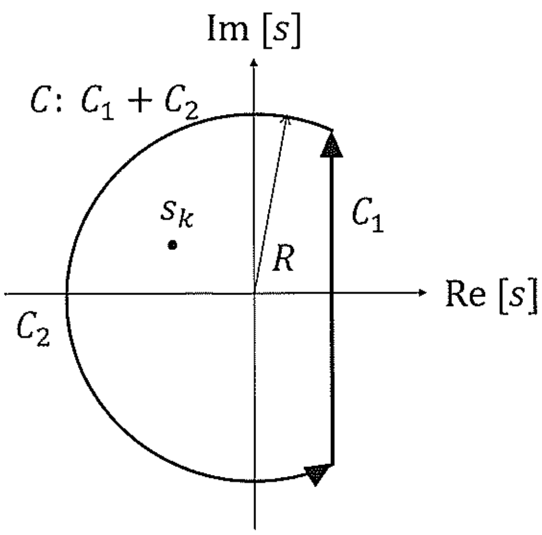
[2-4] Calculate the inverse Laplace transform of the function $G(s)$ obtained from [2-3] based on Eq. (2) using the residue theorem. When there are poles s_k $(k = 1, \dots, n)$ inside the simple closed curve C and the function is regular inside the area covered by C and along C except for the poles, the residue theorem suggests that the loop integral of $H(s)$ along the curve, indicated in the figure below, can be obtained as follows.

$$\oint_C H(s)\, ds = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\,[H, s_k]$$

Furthermore, a residue $\mathrm{Res}\,[H, s_k]$ of $H(s)$ can be calculated as follows at the m^{th} order pole (m is a positive integer) at s_k.

$$\mathrm{Res}\,[H, s_k] = \frac{1}{(m-1)\,!} \lim_{s \to s_k} \frac{d^{m-1}}{ds^{m-1}} \{(s - s_k)^m\, H(s)\}$$

Here it can be assumed that the integral along the curve C_2 is 0 when R becomes infinity in the figure below.



[3] Answer the following questions.

[3-1] Consider two independent events A and B. Given that $P(A \cap B) = 2/25$ and $P(A \cup B) = 13/25$ hold, find $P(A)$ and $P(B)$, where $P(A) < P(B)$.

[3-2] Suppose the probability density function of a continuous random variable X is given by the following equation, where x is a real number and C is a constant. Answer the following questions.

$$f(x) = \begin{cases} C(1 - x^2) & (-1 \leq x \leq 1) \\ 0 & (x < -1,\ x > 1) \end{cases}$$

1) Find the value of the constant C .

2) Find the probability $P\left(-\frac{1}{2} \leq X \leq \frac{1}{2}\right)$.

3) Find the mean $E[X]$ and the variance $V[X]$.