[1] Consider the following equation,

$$\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F} \tag{1}$$

where $\boldsymbol{r}$ is the position vector in the two-dimensional plane, $\boldsymbol{F} = -\frac{1}{r^2}\boldsymbol{e}$, $\boldsymbol{r} = r\boldsymbol{e}$, $|\boldsymbol{e}| = 1$, $r \neq 0$.

[1-1] Suppose $\boldsymbol{F} = \nabla\phi$, where ϕ is the scalar value, and the radial components of $\boldsymbol{F}$ and $\nabla\phi$ in the polar coordinate system are expressed as $-\frac{1}{r^2}$ and $\frac{\partial\phi}{\partial r}$, respectively. Find ϕ.

[1-2] Let the position vectors of points A and B be $\boldsymbol{a}$ and $\boldsymbol{b}$. Here, $|\boldsymbol{a}| = a \neq 0$, $|\boldsymbol{b}| = b \neq 0$, $a \neq b$. Express the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ along the path C from A to B using a and b. The path C can be chosen freely by yourself. Also, calculate the circular integral $\oint \boldsymbol{F} \cdot d\boldsymbol{r}$ by taking a different path from C that returns to point A.

[1-3] Let $d\boldsymbol{r}/dt$ at the points A and B be $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$, respectively. Here, $|\boldsymbol{v}_a| = v_a$, $|\boldsymbol{v}_b| = v_b$. Express the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ along the path C from A to B using v_a and v_b. The following equation can be used,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \frac{d^2\boldsymbol{r}}{dt^2} \cdot d\boldsymbol{r} = \int_C \left(\frac{d^2\boldsymbol{r}}{dt^2} \cdot \frac{d\boldsymbol{r}}{dt}\right) dt.$$

[1-4] Taking the cross products with $\boldsymbol{r}$ on both sides of the equation (1), the following equation can be obtained,

$$\boldsymbol{r} \times \frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{0}.$$

Based on this equation, prove the following equation,

$$\boldsymbol{r} \times \frac{d\boldsymbol{r}}{dt} = \boldsymbol{c}$$

where $\boldsymbol{c}$ is a constant vector.

[2] Answer the following questions when the Laplace transform of function $f(t)$ is defined as follows.

$$L(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

[2-1] Calculate the Laplace transform $L_1(s)$ of the function $f_1(t) = e^{at}$. Note that a is a real number $(a < \mathrm{Re}[s])$.

[2-2] Calculate the Laplace transform $L_2(s)$ of the function $f_2(t) = te^{at}$. Note that a is a real number $(a < \mathrm{Re}[s])$.

[2-3] Find the pole of $L_1(s)$ and show it in the complex plane with respect to s.

[2-4] Prove that the Laplace transform $L_3(s)$ of the function $f_3(t) = e^{at} \sin bt$ is $\frac{b}{(s-a)^2+b^2}$.

Also, answer in what region in the complex plane the poles of $L_3(s)$ must exist for which the function $f_3(t)$ converges to a finite value when $t \to \infty$. Note that a and b are real numbers $(a < \mathrm{Re}[s])$.

[2-5] Consider the Laplace transform $L_g(s)$ of the function $g(t) = \sum_{k=1}^{N} A_k\, e^{a_k t} \sin(b_k t + \phi_k)$. Answer in what region in the complex plane the poles of $L_g(s)$ must exist for which the function $g(t)$ converges to a finite value when $t \to \infty$. Note that a_k, b_k, ϕ_k, and A_k are real numbers $(\max(a_k) < \mathrm{Re}[s])$, and N is a natural number.

[2-6] When the function $u(t)$ satisfies the following ordinary differential equation, answer the range of a real number p for which the function $u(t)$ converges to a finite value when $t \to \infty$.

$$\frac{d^2u}{dt^2} + \frac{du}{dt} + pu = \sin 2t$$