[1] Consider Point P, which is not the origin. Let $\boldsymbol{r} = (x, y, z)$ be the position vector of Point P directed from the origin, $R = |\boldsymbol{r}| = (x^2 + y^2 + z^2)^{1/2}$ and $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Also, let m be a real number. Answer the following questions, in which $\boldsymbol{a} \cdot \boldsymbol{b}$ means the scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

[1-1] Calculate $\nabla \cdot \boldsymbol{r}$.

[1-2] Calculate $\frac{\partial R}{\partial x}$.

[1-3] Describe ∇R by using $\boldsymbol{r}$ and R.

[1-4] Describe ∇R^m by using m, R and ∇R.

[1-5] Determine the blank in the equation below with an appropriate mathematical expression using $\nabla, R, \boldsymbol{r}$ and m.

$$\nabla \cdot (R^m \boldsymbol{r}) = R^m \nabla \cdot \boldsymbol{r} + \boxed{}.$$

[1-6] Based on the above results, describe $\nabla \cdot (R^m \boldsymbol{r})$ by using R and m.

[2] Answer the following questions for the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

[2-1] Find all eigenvalues and their corresponding eigenvectors of the matrix $\boldsymbol{A}$.

[2-2] Let $\boldsymbol{D}$ be a 2×2 diagonal matrix. Based on the results of [2-1], diagonalize the matrix $\boldsymbol{A}$ by finding the matrices $\boldsymbol{P}$ and $\boldsymbol{P}^{-1}$ which satisfy $\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ where $\boldsymbol{P}$ is a regular matrix and $\boldsymbol{P}^{-1}$ is its inverse matrix.

[2-3] Let n be a natural number. For an $n \times n$ matrix $\boldsymbol{B}$, its exponential function $e^{\boldsymbol{B}}$ is defined as

$$e^{\boldsymbol{B}} = \sum_{k=0}^{\infty} \frac{\boldsymbol{B}^k}{k!},$$

where $\boldsymbol{B}^0$ is the $n \times n$ identity matrix.

Derive $e^{\boldsymbol{C}} = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$ for a diagonal matrix $\boldsymbol{C} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where a and b are non-zero real numbers. The following Maclaurin expansion of the exponential function may be used.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

[2-4] For the matrices $\boldsymbol{A}, \boldsymbol{P}$ and $\boldsymbol{P}^{-1}$, a formula $e^{\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}} = \boldsymbol{P}^{-1}e^{\boldsymbol{A}}\boldsymbol{P}$ holds. Calculate $e^{\boldsymbol{A}}$ using the formula and the equation derived in [2-3].

[3] The Fourier transform $F(\omega)$ of a piecewise continuous function $f(x)$ is defined by

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \ ,$$

where i is the imaginary unit, x and ω are real numbers. Answer the following questions.

[3-1] Find the Fourier transform $F(\omega)$ of the function $f(x)$ defined by

$$f(x) = \begin{cases} c & (\,|x| \leq c\,) \\ 0 & (\,|x| > c\,) \end{cases} ,$$

where c is a positive real number.

[3-2] Find all the singular points and their corresponding residues of $g(z)$ defined by the following complex function,

$$g(z) = \frac{1}{z^2 + 4} e^{i\omega z} ,$$

where z is a complex number. Note that when a point a is a pole of order 1 of $g(z)$, the residue $\mathrm{Res}[g, a]$ is obtained by

$$\mathrm{Res}[g, a] = \lim_{z \to a} [(z - a) g(z)] \ .$$

[3-3] Using the residue theorem and the result of [3-2], find the Fourier transform $F(\omega)$ of the function $f(x)$ defined by

$$f(x) = \frac{1}{x^2 + 4} ,$$

where ω is a positive real number. Note that based on the residue theorem, a counterclockwise contour integral of $g(z)$ is expressed as

$$\oint_C g(z)\,dz = 2\pi i\ \mathrm{Res}[g, a]\,,$$

where $g(z)$ has a singular point of a inside a closed curve C. The function is regular analytic inside the closed curve C except at the point a and continuous along C.

Furthermore, the following Jordan's lemma may be used.

$$\lim_{R\to\infty}\left|\int_\Gamma h(z)e^{ibz}dz\right| \to 0\,,$$

where b is a positive real number and $h(z)$ is a rational function in which the degree of the denominator is at least one greater than that of the numerator. The integral path Γ is a counterclockwise upper semicircle with the radius R centered at the origin of the complex plane.