[1] Answer the following questions regarding the Fourier transform. Here, the Fourier transform of $f(x)$ is defined as $F(y)$ and expressed as follows:

$$F(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixy} dx.$$

In addition, the inverse Fourier transform of $F(y)$ is defined as $f(x)$ and expressed as follows:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(y) e^{ixy} dy.$$

Here, $i^2 = -1$.

[1-1] Derive the Fourier transform of

$$f(x) = e^{-|x|}.$$

[1-2] Using the result of [1-1], calculate the following definite integral

$$\int_{0}^{\infty} \frac{\cos x}{1 + x^2}\, dx.$$

[2] Answer the following questions.

[2-1] Answer the following questions for the below matrix

$$A = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix} \quad (0 < p < 1,\ 0 < q < 1)\,.$$

1) Find the eigenvalues and the eigenvectors.

2) Assuming the two eigenvectors derived in 1) as P_1 and P_2, and the square matrix consisting of P_1 and P_2 as $P = (P_1, P_2)$, calculate $P^{-1}AP$.

3) Calculate A^n, letting n be a positive integer.

4) When $n \to \infty$, express the matrix A^n.

[2-2] Consider the orthogonal linear coordinate system O-xyz, where O is the origin, and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the positive unit vectors in the respective directions of x, y, z axes.

Answer the following questions for the below vector field $\boldsymbol{B}$

$$\boldsymbol{B} = (2p+1)x^2y\boldsymbol{i} + \{px^3 + (1+p)yz\}\boldsymbol{j} + (y^2 + pz^2)\boldsymbol{k}$$

where p is a constant.

1) Calculate the divergence of the vector field $\boldsymbol{B}$, $\operatorname{div}\boldsymbol{B}$.

2) Determine the constant value p, so that the rotation of the vector field $\boldsymbol{B}$, $\operatorname{rot}\boldsymbol{B}$, becomes identically a zero vector $\boldsymbol{O}$.

[3] Let a random variable X represent the number of years, in which river water level exceeds the height of a riverbank levee and the overflow happens, in 100 years at a given river reach with levees on both sides. The overflow can occur once a year with annual maximum water level. Let p be the probability of annual maximum water level exceeding the top of levee, and X follows binomial distribution $\mathrm{B}(100, p)$. Answer the following questions.

[3-1] Express the probability of non-occurrence $\mathrm{Pr}[X = 0]$ of the overflow in 100 years using p.

[3-2] Suppose that the probability of 1 or more overflow occurrence in 100 years is less than or equal to 1/100. Show this relation by an inequality using p and 1/100.

[3-3] When annual maximum water level h has the probability density function $f(h)$ as in the following figure, express the exceedance probability of h, $W(h) = \int_h^{\infty} f(k)dk$ (here, $2\ \mathrm{m} \leq h \leq 4\ \mathrm{m}$).